Calculus I (for CS)

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This is an unofficial lecture organized by S.V. Cover.

Current status

Since these slides are the result of free time work, they are not yet complete. The following topics are currently missing from the slides:

- More on variation of parameters (also for first order DEs)
- Implicit differentiation
- Improper integrals
- Logarithmic differentiation (although something similar is discussed)
- Trigonometric integrals
- Rolle's theorem, Mean Value Theorem, Intermediate Value Theorem, Fundamental Theorem of Calculus, ...
- Derivative tests
- Arc length of a curve
- ... possibly more??

Still, we hope the slides are helpful.

1. Limits

2. Complex numbers

3. Differentiation & integration
   - Differentiation
   - Integration & Applications

4. Differential equations (DEs)
   - First order differential equations
   - Second order differential equations

5. Taylor polynomials

Basic limits

- $\lim\limits_{x\to1}\frac{x^2-1}{x-1}=\lim\limits_{x\to1}\frac{(x+1)(x-1)}{(x-1)}=\lim\limits_{x\to1}(x+1)=1+1=2$
- $\lim\limits_{t\to-4^-}\frac{x^2+9x+20}{|x+4|}=\lim\limits_{t\to-4^-}\frac{x^2+9x+20}{-(x+4)}=\lim\limits_{t\to-4^-}\frac{(x+5)(x+4)}{-(x+4)}=\lim\limits_{t\to-4^-}-(x+5)=-(-4+5)=-1$
- $\lim\limits_{x\to-4}\frac{\sqrt{x^2+9}-5}{x+4}=\lim\limits_{x\to-4}\frac{\sqrt{x^2+9}-5}{x+4}\cdot\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}=\lim\limits_{x\to-4}\frac{x^2+9-25}{(x+4)(\sqrt{x^2+9}+5)}=\lim\limits_{x\to-4}\frac{(x+4)(x-4)}{(x+4)(\sqrt{x^2+9}+5)}=\lim\limits_{x\to-4}\frac{x-4}{\sqrt{x^2+9}+5}=\frac{-4-4}{\sqrt{(-4)^2+9}+5}=-\frac{4}{5}$

L'Hôpital's rule

L'Hôpital's rule

If we have a limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ **that is an indeterminate form of type $\mathbf{\frac{0}{0}}$ or $\mathbf{\frac{\infty}{\infty}}$**, then[a], we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

[a] if f and g are differentiable and $g'(x) \neq 0$ in a neighborhood of a (except possibly at a) and if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or is $\pm\infty$

- $\lim_{x \to 0} \frac{2 \sin x - \sin 2x}{x - \sin x} = \lim_{x \to 0} \frac{2 \cos x - 2 \cos 2x}{1 - \cos x} = \lim_{x \to 0} \frac{-2 \sin x + 4 \sin 2x}{\sin x} = \lim_{x \to 0} \frac{-2 \cos x + 8 \cos 2x}{\cos x} = \frac{-2+8}{1} = \boxed{6}$

Squeeze theorem

Squeeze theorem

If $f(x) \leq g(x) \leq h(x)$ for x near a (except possibly at a), then:

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \quad \Longrightarrow \quad \lim_{x \to a} g(x) = L$$

- $\lim_{x \to 0} x^2 \sin \frac{1}{x} = ?$
- We have $-1 \leq \sin \frac{1}{x} \leq 1$ for all real non-zero x, and $x^2 \geq 0$, thus also: $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$
- $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$, therefore $\boxed{\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0}$.

Limits with e

Euler's number

Euler's number is equal to $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

- Question: calculate $\lim_{x\to\infty} \left(\frac{x+11}{x+5}\right)^{7x+3}$.
- Solution: next slide.

Example limit with e

$$\lim_{x\to\infty}\left(\frac{x+11}{x+5}\right)^{7x+3} = \lim_{x\to\infty}\left(1+\frac{6}{x+5}\right)^{7x+3} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{7x+3}$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{\left(\frac{x}{6}+\frac{5}{6}\right)\cdot 6\cdot 7-35+3} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{\left(\frac{x}{6}+\frac{5}{6}\right)\cdot 42-32}$$

$$= \lim_{x\to\infty}\left(\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{\left(\frac{x}{6}+\frac{5}{6}\right)}\right)^{42}\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{-32}$$

$$= \left(\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{\left(\frac{x}{6}+\frac{5}{6}\right)}\right)^{42}\cdot 1 = \boxed{e^{42}}$$

Basic arithmetic

Complex numbers ($a + bi$ with $a, b \in \mathbb{R}$) behave just like you'd expect when doing simple arithmetic. For example:

$(12 - 23i) + (3 + 6i) = 15 - 17i$

$(12 - 23i) - (3 + 6i) = 9 - 29i$

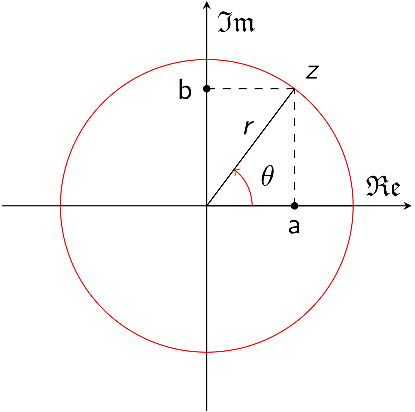
$(5 + 6i)(7 + 8i) = 35 + 82i + 48i^2 = -13 + 82i$

For division, use this trick with the complex conjugate of the denominator:

$$\frac{2-3i}{4-5i} = \frac{2-3i}{4-5i} \cdot \underbrace{\frac{4+5i}{4+5i}}_{=1} = \frac{(2-3i)(4+5i)}{(4-5i)(4+5i)} = \frac{23-2i}{41} = \boxed{\frac{23}{41} - \frac{2}{41}i}$$

Complex plane

Complex numbers lie in the *complex plane*.



$$z = a + bi = re^{i\theta} = r[\cos\theta + i\sin\theta]$$

Instead of writing $a + bi$ (rectangular form), we can equivalently write $re^{i\theta}$ or $r[\cos\theta + i\sin\theta]$. The latter is known as the *polar form*.

Converting to exponential and polar form

- The modulus of a complex number $z = x + yi$ is $\boxed{r = |z| = \sqrt{x^2 + y^2}}$.
- In order to find the (principal) argument $\text{Arg}(z)$, use the formula:

$$\theta = \text{Arg}(z) = \text{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \arctan(\frac{y}{x}) + \pi & x < 0, y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & x < 0, y < 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0 \\ \text{(undefined)} & x = 0, y = 0 \end{cases}$$

This formula gives the angle between $-\pi < \theta \leq \pi$.
Angles are in radians! Fun fact: the function $\text{atan2}(y, x)$ is implemented in many programming languages, including C, Java, ...

Converting to exponential form (examples)

- $1 + i = \sqrt{2} \cdot e^{i\pi/4}$ (since $\sqrt{1^2 + 1^2} = \sqrt{2}$ and $\text{Arg}(1 + i) = \frac{\pi}{4}$)
- $1 - i = \sqrt{2} \cdot e^{-i\pi/4}$ (now the angle is $\text{Arg}(1 - i) = -\frac{\pi}{4}$)
- $-10 = 10e^{i\pi}$ (-10 lies on the negative real axis, so the angle is π)

Multiplying two complex numbers (polar form)

- Suppose that we have two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Let's see what happens when we multiply:
- $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$.
- Thus, when multiplying two complex numbers, the modulus gets multiplied and the arguments (angles) get added.

Getting the nth roots of a number

- Question: find all $z \in \mathbb{C}$ for which $z^5 = -10$.
- Solution: we have that $arg(-10) = \pi + 2k\pi$ and $|-10| = 10$. So we can write $z^5 = -10 = 10e^{i(\pi+2k\pi)}$. (for $k \in \mathbb{Z}$)
- Raising left-hand and right-hand side to the power $\frac{1}{5}$, we obtain $z = 10^{1/5}e^{i(\frac{\pi}{5}+\frac{2k\pi}{5})}$
- We have five solutions, so for example, take k to be $0, 1, 2, 3, 4$ to find the following solutions:

$$z = 10^{1/5}e^{i(\frac{\pi}{5})} \quad \vee \quad z = 10^{1/5}e^{i(\frac{3\pi}{5})}$$
$$\vee \quad z = 10^{1/5}e^{i\pi} = -10^{1/5} \quad \vee \quad z = 10^{1/5}e^{i(\frac{7\pi}{5})} \quad \vee \quad z = 10^{1/5}e^{i(\frac{9\pi}{5})}$$

- (These are all solutions: if we were to go on for $k = 5, 6, \ldots$, then the solutions would repeat because sine and cosine (and thus, $e^{i\theta}$) have a period of 2π.)

De Moivre's formula

- Question: write $(\sqrt{3}+i)^{1000}$ in the form $a+bi$
- One approach would be to expand brackets a thousand times. However, there is a faster method.
- We can write $(\sqrt{3}+i)^{1000} = (2e^{i\pi/6})^{1000} = 2^{1000}e^{1000i\pi/6} = 2^{1000}e^{4i\pi/6} = 2^{1000}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \boxed{-2^{999}+2^{999}\sqrt{3}i}$

De Moivre's formula

Suppose we have a complex number $z = e^{i\theta}$. Then, as in the example, $z^n = (e^{i\theta})^n = e^{in\theta}$. Rewriting in polar form gives us *De Moivre's formula*:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

In practice, using the exponential form (as in the example) may be easier.

Basic differentiation

- Differentiation is usually easy, thus we will only include some examples without explaining all rules first. (One exception is differentiating x^x which will be covered in more detail.)
- Example: $[\ln(2x^2 - 3x + 4)]' = \frac{[2x^2-3x+4]'}{2x^2-3x+4} = \frac{4x-3}{x^2-3x+4}$
- Example: $\frac{d}{dx}(3) = 0$
- Example: $(2x+1)^7 = 14(2x+1)^6$ (do not expand brackets, but use the chain rule instead)
- Example: $\frac{d^4(\sin x)}{dx^4} = \frac{d^3(\cos x)}{dx^3} = \frac{d^2(-\sin x)}{dx^2} = \frac{d(-\cos x)}{dx} = \sin x$
- Therefore also:
  $\sin x = \frac{d^4(\sin x)}{dx^4} = \frac{d^8(\sin x)}{dx^8} = \frac{d^{12}(\sin x)}{dx^{12}} = \ldots = \frac{d^{400}(\sin x)}{dx^{400}} = \ldots$
  (This is useful when computing the Taylor series of $\sin x$.)

Differentiation example

$$\frac{d}{dx}\sin\left(\frac{e^{x^3}}{3x^2}\right) = \left[\frac{e^{x^3}}{3x^2}\right]'\cos\left(\frac{e^{x^3}}{3x^2}\right) = \frac{3x^2[e^{x^3}]' - e^{x^3}[3x^2]'}{(3x^2)^2}\cos\left(\frac{e^{x^3}}{3x^2}\right)$$

$$= \frac{3x^2(e^{x^3}[x^3]') - 6xe^{x^3}}{(3x^2)^2}\cos\left(\frac{e^{x^3}}{3x^2}\right)$$

$$= \frac{3x^2(3x^2e^{x^3}) - 6xe^{x^3}}{(3x^2)^2}\cos\left(\frac{e^{x^3}}{3x^2}\right)$$

$$= \boxed{e^{x^3}\left(1 - \frac{2}{3x^3}\right)\cos\left(\frac{e^{x^3}}{3x^2}\right)}$$

A trick to differentiate x^x

- We know how to differentiate $[c^x]' = c^x \ln c$ as well as $[x^c]' = cx^{c-1}$. But what if x appears in both the base and the exponent?
- Let $y(x) = x^x$. Then $y = (e^{\ln x})^x = e^{x \ln x}$
  The derivative then becomes: $y'(x) = e^{x \ln x} \cdot [x \ln x]'$
  Which is equal to $\boxed{y'(x) = x^x(1 + \ln x)}$
- Alternatively, use the following method: $y = x^x$
  Taking the logarithm: $\ln y = \ln(x^x) = x \ln x$
  Implicit differentiation: $\frac{y'}{y} = 1 + \ln x$ (Why not $\frac{1}{y}$?)
  Multiply both sides with $y = x^x$: $\boxed{y'(x) = x^x(1 + \ln x)}$
- Analogous reasoning can be used to differentiate similar functions like $(2x)^{5x+1}$.

Differentiating $f(x)^{g(x)}$

We could create a "power rule" for functions, similar to the product rule, quotient rule etc.
So, say we have two functions $f = f(x)$ and $g = g(x)$, and that $y(x) = f(x)^{g(x)}$ Then we have:

$$y = f^g = (e^{\ln f})^g = e^{g \ln f}$$
$$y' = e^{g \ln f}[g \ln f]' = f^g [g \ln f]'$$
$$y' = f^g \left(g \frac{f'}{f} + g' \ln f \right)$$

So our "power rule" turns out to be

$$\boxed{[f^g]' = f^g \left(g \frac{f'}{f} + g' \ln f \right)}$$

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Integration: simple integrals

- $\int x dx = \frac{1}{2}x^2 + C$
- $\int \cos(x) dx = \sin(x) + C$
- $\int_2^3 (\frac{1}{x} + \frac{1}{x^2}) dx = [\ln|x| - \frac{1}{x}]_2^3 = (\ln(3) - \frac{1}{3}) - (\ln(2) - \frac{1}{2}) = \ln(\frac{3}{2}) + \frac{1}{6}$
- Do not forget to write $+C$ for indefinite integrals!

Integration basics

- Sometimes, you see both a function and a function's derivative in the same integral. Sometimes you can then use the chain rule in the opposite direction, as follows:
- $\int \frac{\cos x}{\sin x} dx = \int \frac{1}{\sin x} [\sin x]' dx = \ln|\sin x| + C$
- $\int e^x \sin(100 + 3e^x) dx = \frac{1}{3} \int [100 + 3e^x]' \sin(100 + 3e^x) dx = -\frac{1}{3} \cos(100 + 3e^x) + C$
- $\int \frac{x dx}{x^2+137} = \frac{1}{2} \int \frac{2x dx}{x^2+137} = \frac{1}{2} \int \frac{[x^2+137]' dx}{x^2+137} = \frac{1}{2} \ln\left|x^2 + 137\right| + C$
- These integrals can also be solved using a substitution.

Substitution rule (1)

- The integrals from the last slide can also be solved using substitution.
- Example: $\int \frac{xdx}{x^2+137}$
- When we set $u = x^2 + 137$, we find $du = 2xdx$, thus:
- $\int \frac{xdx}{x^2+137} = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|x^2 + 137| + C$
- Do not forget to convert the u back to x!

Substitution rule (2)

- Question: calculate $\int \frac{(\ln x)^2}{x} dx$
- Detailed solution: we observe that $[\ln x]' = \frac{1}{x}$, and we also see that $\frac{1}{x}$ appears in our integral. Thus, let us try $u = \ln x$.
- Then $du = \frac{1}{x} dx$, and therefore

$$\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{1}{3} u^3 + C = \boxed{\frac{1}{3} (\ln x)^3 + C}$$

- (It takes practice to find good substitutions.)

Integration by parts (1)

General form – Integration By Parts

Indefinite integrals: $\int u(x)v'(x)dx = [u(x)v(x)] - \int u'(x)v(x)dx$

Definite integrals: $\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$

- $\int x\cos x dx = [x\sin x] - \int 1 \cdot \sin x dx = x\sin x + \cos x + C$
- $\int xe^{3x}dx = [x \cdot \frac{1}{3}e^{3x}] - \int 1 \cdot \frac{1}{3}e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$
- We see from examples 1 and 2 that if we have x in front of something we know how to integrate, then we can also integrate the new thing! However, "I.B.P." is more powerful than that.

Integration by parts (2)

General form – Integration By Parts

Indefinite integrals: $\int u(x)v'(x)dx = [u(x)v(x)] - \int u'(x)v(x)dx$
Definite integrals: $\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$

- Example: $\int \log_2(3w^{w^4})dw = \frac{1}{\ln 2}\int(\ln(3) + w^4\ln(w))dw = \frac{\ln 3}{\ln 2}w + \frac{1}{\ln 2}\int \ln(w)\cdot w^4 dw = \frac{\ln 3}{\ln 2}w + \frac{1}{\ln 2}\left([\ln(w)\cdot\frac{1}{5}w^5] - \int \frac{1}{w}\cdot\frac{1}{5}w^5 dw\right) = \frac{\ln 3}{\ln 2}w + \frac{1}{\ln 2}\left([\frac{1}{5}w^5\ln(w)] - \int \cdot\frac{1}{5}w^4 dw\right) = \frac{\ln 3}{\ln 2}w + \frac{1}{\ln 2}\left(\frac{5}{25}w^5\ln(w) - \frac{1}{25}w^5\right) + C = \boxed{\frac{\ln 3}{\ln 2}w + \frac{1}{25\ln 2}\left(5w^5\ln(w) - w^5\right) + C}$

Repeated integration by parts

- We saw that if we have something we can integrate (say $\sin x$ or e^x), then we can also integrate the product of that function with x (so we can integrate $x \sin x$ or xe^x).
- By applying I.B.P multiple times, we can even work away higher powers of x:
- Question: integrate $\int x^3 \sin(x)dx$.
- Solution (pay attention to plus/minus!):

$$\begin{aligned}
\int x^3 \sin(x)dx &= [-x^3 \cos x] + 3 \int x^2 \cos(x)dx \\
&= [-x^3 \cos x] + 3 \left([x^2 \sin x] - 2 \int x \sin(x)dx \right) \\
&= [-x^3 \cos x] + 3 \left([x^2 \sin x] - 2 \left([-x \cos x] + \int \cos(x)dx \right) \right) \\
&= [-x^3 \cos x] + 3 \left([x^2 \sin x] - 2 \left([-x \cos x] + \int \cos(x)dx \right) \right) \\
&= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C
\end{aligned}$$

Solving $\int e^x \sin(x)dx$ and $\int e^x \cos(x)dx$ with I.B.P.

- Question: find $\int e^x \sin(x)dx$.
- Solution:

$$\int e^x \sin(x)dx = [e^x \sin x] - \int e^x \cos(x)dx$$
$$= e^x \sin x - \left([e^x \cos x] + \int e^x \sin(x)dx\right)$$
$$= e^x \sin x - e^x \cos x - \int e^x \sin(x)dx$$

We can now add $\int e^x \sin(x)dx$ to both sides:

$$2\int e^x \sin(x)dx = e^x(\sin x - \cos x) + C$$

$$\implies \boxed{\int e^x \sin(x)dx = \frac{1}{2}e^x(\sin x - \cos x) + C_*} \quad \left(\text{where } C_* = \frac{1}{2}C\right)$$

Trigonometric substitutions: example (slide 1)

- Question: find

$$\int_2^5 \frac{\sqrt{x^2-4}}{x^3}dx$$

- Solution: we substitute $x = 2\sec\theta$ (for $0 \leq \theta < \pi/2$ or $\pi \leq \theta < 3\pi/2$). It will be explained later why we choose this substitution. Then we have $dx = 2\sec\theta\tan\theta d\theta$.We find:

$$\sqrt{x^2-4} = \sqrt{(2\sec\theta)^2-4} = 2\sqrt{\sec^2\theta-1} = 2\sqrt{\tan^2\theta}$$
$$= 2\left|\tan\theta\right| = 2\tan\theta$$

(We know that $2\left|\tan\theta\right| = 2\tan\theta$ because $\tan\theta \geq 0$ for $0 \leq \theta < \pi/2$ or $\pi \leq \theta < 3\pi/2$)
Integral bounds: if $x = 2\sec\theta = 2$ then $\theta = 0$. If $x = 2\sec\theta = 5$ then $\theta = \arccos(\frac{2}{5})$. The integral will be worked out in the next slide.

Trigonometric substitutions: example (slide 2)

Integral bounds: if $x = 2\sec\theta = 2$ then $\theta = 0$. If $x = 2\sec\theta = 5$ then $\theta = \arccos(\frac{2}{5})$. So, we have:

$$\int_2^5 \frac{\sqrt{x^2-4}}{x^3}dx = \int_0^{\arccos(2/5)} \frac{2\tan\theta}{8\sec^3\theta} 2\sec\theta\tan\theta d\theta$$

$$= \frac{1}{2}\int_0^{\arccos(2/5)} \sin^2\theta d\theta = \frac{1}{2}\int_0^{\arccos(2/5)} \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\arccos(2/5)} = \boxed{\frac{\arccos(\frac{2}{5})}{4} - \frac{1}{8}\sin\left(2\arccos\left(\frac{2}{5}\right)\right)}$$

$$= \frac{\arccos(\frac{2}{5})}{4} - \frac{1}{8}\left(2\sin\left(\arccos\left(\frac{2}{5}\right)\right)\cos\left(\arccos\left(\frac{2}{5}\right)\right)\right)$$

$$= \frac{\arccos(\frac{2}{5})}{4} - \frac{1}{10}\sin\left(\arccos\left(\frac{2}{5}\right)\right) = \boxed{\frac{\arccos(\frac{2}{5})}{4} - \frac{\sqrt{21}}{50}}$$

Trigonometric substitutions table

- In the previous example, we substituted $x = 2\sec\theta$ and everything magically worked out. How did we find this substitution? Well, we used this scheme:

Trigonometric substitutions for integration

Expression	Use the substitution	And use
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Watch out!

- Question: solve
$$\int t\sqrt{t^2+2}dt$$
- Observation: we recognize that this integral contains the term $\sqrt{t^2+2}$, thus we could try to make a trigonometric substitution. However, this is going to cost much time.
  It is much easier to say: $u = t^2 + 2$, so $du = 2tdt$ and solve the integral that way, without any need for "trig sub".
- So, please think twice and be sure that no other way works, before doing trig substitution!

Integration by partial fractions

Sometimes you want to integrate the quotient of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

.

Assume for now (**very important!**) that the degree of P is lower than the degree of Q (otherwise do long division first).

Then we compute the integral as follows:

1. Write $f(x)$ as a *sum* of terms (*partial fractions*) of the form $\frac{A}{(ax+b)^i}$ and $\frac{Ax+B}{(ax^2+bx+c)^j}$ (see next slide).
2. Solve for the constants A, B, ...
3. Integrate each partial fraction.

Finding the partial fraction decomposition

We have $f(x) = \frac{P(x)}{Q(x)}$ (where $\deg(P) < \deg(Q)$). The goal is to write $f(x)$ as a sum of partial fractions.

1. If $Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_kx + b_k)$ is the product of distinct linear factors, then there exist constants $A_1, \dots, A_k$ s.t. $\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_k}{a_kx+b_k}$.
2. If some linear factor is repeated, it will occur multiple times in the partial fraction decomposition: if we have $Q(x) = \dots \cdot (a_ix + b_i)^r \cdot \dots$, we get $\frac{P(x)}{Q(x)} = \dots + \left(\frac{A_1}{a_ix+b_i} + \frac{A_2}{(a_ix+b_i)^2} + \dots + \frac{A_r}{(a_ix+b_i)^r}\right) + \dots$.
3. A distinct **irreducible** quadratic factor $a_ix^2 + b_ix + c_i$ of $Q(x)$ adds a term like $\frac{Ax+B}{a_ix^2+b_ix+c_i}$ to the partial fraction decomposition.
4. A repeated irreducible quadratic factor $(a_ix^2 + b_ix + c)^r$ will give us $\frac{A_1x+B_1}{a_ix^2+b_ix+c} + \frac{A_2x+B_2}{(a_ix^2+b_ix+c)^2} + \dots + \frac{A_rx+B_r}{(a_ix^2+b_ix+c)^r}$ in the decomposition.

Common mistake: forgetting the $+B$ in cases 3 and 4!

Finding the partial fraction decomposition: examples

Examples illustrating the last slide:

- $f(x) = \frac{3x+6}{(3x+4)(3x+5)} \quad \rightarrow \quad f(x) = \frac{A}{3x+4} + \frac{B}{3x+5}$
- $f(x) = \frac{x^2+42x+42}{(x+1)(2x+42)^2} \quad \rightarrow \quad f(x) = \frac{A}{x+1} + \frac{B_1}{2x+42} + \frac{B_2}{(2x+42)^2}$
- $f(x) = \frac{3x+6}{(3x+4)(x^2+3x+4)^3} \quad \rightarrow \quad f(x) = \frac{A}{3x+4} + \frac{B_1}{x^2+3x+4} + \frac{B_2}{(x^2+3x+4)^2} + \frac{B_3}{(x^2+3x+4)^3}$

But:

- $f(x) = \frac{1}{(2x+4)(x+3)(x+2)} \quad \rightarrow \quad f(x) = \frac{A}{2x+4} + \frac{B}{x+3} + \frac{C}{x+2}$???
  NO: $2x+4$ is a constant multiple of $x+2$ so the correct decomposition is $f(x) = \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2} + \frac{B}{x+3}$.
- $f(x) = \frac{3x+6}{(3x+4)(x^2+5x+4)^2} \quad \rightarrow \quad f(x) = \frac{A}{3x+4} + \frac{B_1}{x^2+5x+4} + \frac{B_2}{(x^2+5x+4)^2}$???
  NO: $x^2+5x+4 = (x+4)(x+1)$ is not an *irreducible* quadratic!
- $f(x) = \frac{x^2+x+1}{(x+1)(x+2)} \quad \rightarrow$ STOP! The degree of the numerator is not less than the degree of the denominator, so we must divide first.

Long division before setting up P.F.D.

If we have some fraction $f(x) = \frac{P(x)}{Q(x)}$ where $\deg(P) \geq \deg(Q)$, then we must divide first.
For example:

$$f(x) = \frac{2x^3 - x^2 - 8x - 3}{(x+1)(x-2)} = \frac{2x(x^2 - x - 2) + x^2 - 4x - 3}{x^2 - x - 2}$$
$$= 2x + \frac{(x^2 - x - 2) - 3x - 1}{x^2 - x - 2} = 2x + 1 - \frac{3x+1}{(x+1)(x-2)}$$

(The $2x + 1$ is easy to integrate, and we can do partial fractions to integrate $\frac{3x+1}{(x+1)(x-2)}$.)

Tips for partial fractions

After having set up the partial fraction decomposition **and solved for its constants** (example on next slide), we should integrate each partial fraction individually.

The following guidelines are useful:

- $\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C$ (the most common one)
- $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan(\frac{x}{a}) + C$
- integrate $\frac{Ax+B}{ax^2+bx+c}$ (where $b^2 - 4ac < 0$) by completing the square in the denominator and substituting to get $\int \frac{Cu+D}{u^2+a^2}\,du = \int C\frac{u}{u^2+a^2}\,du + D\int \frac{1}{u^2+a^2}\,du$.
- integrate $\int \frac{1}{(x^2+a^2)^2}\,dx$ by substituting $x = a\tan\theta$.

Example on partial fractions

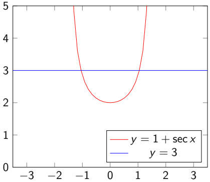
Sorry, not enough time to write it out.
So let's do one of the following on the board:

- $\int \frac{1}{(x+2)(x-3)}\,dx$ (easy)
- $\int \frac{1}{x-\sqrt[5]{x}}\,dx$ (medium)
- $\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx$ (will take the rest of the lecture)

Volumes of solids of revolution ('normal' way)

Question: compute the volume of the solid obtained by rotating the region bounded by the curves $y = 1 + \sec x$ and $y = 3$ about the line $y = 1$.

Solution: the curves intersect when $1 + \sec x = 3$, i.e. when $\cos x = \frac{1}{2}$. We just take the solutions $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$.
Then we can compute the volume V as follows:



$$V = \int_{-\pi/3}^{\pi/3} \left(\pi(3-1)^2 - \pi(1+\sec x - 1)^2\right)\, dx = \pi \int_{-\pi/3}^{\pi/3} (4 - \sec^2 x)\, dx$$

$$= \pi\,[4x - \tan x]_{-\pi/3}^{\pi/3} = \pi(4\pi/3 - \sqrt{3}) - \pi(-4\pi/3 + \sqrt{3}) = \boxed{\frac{8\pi^2}{3} - 2\pi\sqrt{3}.}$$

(Note: we could have used symmetry to make the integral easier.)

Volumes of solids of revolution (cylindrical shells)

Sometimes it is (much) easier to compute volumes of solids of revolution as follows.

Question: find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = \sqrt{5 + x^2}$, $y = 0$, $x = 0$ and $x = 2$ around the y-axis.

Solution: on the board (answer is $\frac{2}{3}\pi(27 - 5^{3/2})$).

Sample question (slide 1)

Let $f(x) = 2xe^{2x}$.

- (a) Find the x- and y-coordinates of the local minima and maxima of $f(x)$.
- (b) Find the range of $f(x)$ for $-1 \leq x \leq 2$.
- (c) Find the area between the x-axis and the graph of $f(x)$ for $-1 \leq x \leq 2$.

Sample question (slide 2)

Let $f(x) = 2xe^{2x}$.

- (question a) Find the x- and y-coordinates of the local extrema of $f(x)$.
- Solution: we compute the derivative $f'(x) = 2e^{2x} + 4xe^{2x}$ and set it to zero, so

$$2e^{2x} + 4xe^{2x} = 0$$
$$2e^{2x}(1 + 2x) = 0$$

Since $2e^{2x}$ is never equal to zero, our only solution to $f'(x) = 0$ is $x = -\frac{1}{2}$.
The corresponding y-coordinate is $f(-\frac{1}{2}) = 2(-\frac{1}{2}) \cdot e^{2(-\frac{1}{2})} = -\frac{1}{e}$.
We see that $f'(-1) = -2e^{-2} < 0$ and $f'(0) = 2 > 0$, so (by the first derivative test) we have a minimum, with coordinates $\boxed{\left(-\frac{1}{2}, -\frac{1}{e}\right)}$.

Sample question (slide 3)

Let $f(x) = 2xe^{2x}$.

- (question b) Find the range of $f(x)$ for $-1 \leq x \leq 2$.
- Solution: from question (a), we know that this function has a minimum with the coordinates $\left(-\frac{1}{2}, -\frac{1}{e}\right)$. This minimum lies within the domain $-1 \leq x \leq 2$. We are also interested in the value of $f(x)$ at the bounds of the domain, so we compute $f(-1) = -2e^{-2}$ and $f(2) = 4e^4$. We observe that $-\frac{1}{e} < -2e^{-2} < 4e^4$, so the range for $-1 \leq x \leq 2$ is $\boxed{\left[-\frac{1}{e}, 4e^4\right]}$.

Sample question (slide 4)

Let $f(x) = 2xe^{2x}$.

- (question c) Find the area between the x-axis and the graph of $f(x)$ for $-1 \leq x \leq 2$.
- Solution: first compute the antiderivative of $f(x)$ using integration by parts:

$$\int f(x)dx = \int 2xe^{2x}dx = xe^{2x} - \int e^{2x}dx = \left(x - \frac{1}{2}\right)e^{2x} + C$$

We see that $f(x)$ is negative for $x < 0$ and positive for $x > 0$, so we have to split up the integral (we do not want "negative area"). Our answer then becomes

$$\left|\int_{-1}^{0} f(x)dx\right| + \left|\int_{0}^{2} f(x)dx\right| = \left|-\frac{1}{2} + \frac{3}{2}e^{-2}\right| + \left|\frac{3}{2}e^{4} + \frac{1}{2}\right| = \boxed{1 + \frac{3}{2}\left(e^{4} - e^{-2}\right)}$$

Linear first order differential equations

Linear first order DE

In order to solve the linear differential equation

$$y' + P(x)y = Q(x)$$

multiply both sides by $e^{\int P(x)dx}$ (the integrating factor), rewrite using the product rule for derivatives, and integrate both sides.

- Question: solve $y' + 2xy = x$
- Solution: next slide.

Example: linear 1st order DEs

- Question: solve $y' + 2xy = x$
- Solution: the integrating factor is $e^{\int 2x dx} = e^{x^2}$ (the constant of integration in the exponent is omitted). So we multiply both sides of the DE by e^{x^2} and obtain

$$e^{x^2} y' + 2xe^{x^2} y = xe^{x^2}$$

Using the product rule for derivatives, this can be rewritten as

$$\left[e^{x^2} y\right]' = xe^{x^2}$$

We integrate both sides and obtain

$$e^{x^2} y = \int xe^{x^2} dx = \frac{1}{2} e^{x^2} + C$$

We can divide both sides by e^{x^2} to find the solution $\boxed{y = \frac{1}{2} + Ce^{-x^2}}$

Separable first order differential equations

- A **separable** first order differential equation is an equation where the x's and y's can be "separated", thus the equation can be rewritten into the form $y'f(y) = g(x)$. The method to solve these is to rewrite that equation to the form $f(y)dy = g(x)dx$ and then integrate both sides $\int f(y)dy = \int g(x)dx$:
- Question: $xyy' = x^2 + 1$
- Solution: rewrite the equation into $ydy = \frac{x^2+1}{x}dx$ and integrate both sides: $\int ydy = \int \frac{x^2+1}{x}dx$
  So we find $\frac{1}{2}y^2 = \frac{1}{2}x^2 + \ln|x| + C$
  So the final answer becomes $\boxed{y = \pm\sqrt{x^2 + 2\ln|x| + C_*}}$ where $C_* = 2C$.

(Linear) second order differential equations

General form

Homogeneous: $ay'' + by' + cy = 0$
Non-homogeneous: $ay'' + by' + cy = f(x)$
(here we will only deal with the case that a, b and c are constant real numbers)

Example:

$$y'' - 8y' + 15y = 0$$

Solving a homogeneous 2nd order DE: basic example

Example: $y'' - 8y' + 15y = 0$
First construct the corresponding quadratic equation and solve it:

$$r^2 - 8r + 15 = 0$$
$$(r-3)(r-5) = 0$$
$$r = 3 \vee r = 5$$

We have two distinct real roots (3 and 5), so the general solution of the DE is

$$\boxed{y = c_1 e^{3x} + c_2 e^{5x}}$$ for any constants c_1 and c_2

“Algorithm” to solve the homogeneous case

You have some DE which you want to solve: $\boldsymbol{ay'' + by' + cy = 0}$

- Step 1: construct the characteristic equation: $ar^2 + br + c = 0$
- Step 2: solve it (compute the roots/solutions r_1 and r_2)
- Step 3: use the following scheme to find your final answer:

Solution cases for homogeneous 2nd order DE

- Real (non-equal) roots: $y_{(x)} = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
- One real root: $y_{(x)} = c_1 e^{rx} + c_2 x e^{rx}$
- Case $r_{1,2} = \alpha \pm i\beta$: $y_{(x)} = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$

Solving a homogeneous 2nd order DE: another example

Example: $7y'' - 7y' + 2y = 0$

First construct the corresponding quadratic equation and solve it:

$$7r^2 - 7r + 2 = 0$$

$$r_{1,2} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 7 \cdot 2}}{2 \cdot 7}$$

$$r_{1,2} = \frac{7 \pm \sqrt{-7}}{14}$$

$$r_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{7}}{14}$$

(Recall: Case $r_{1,2} = \alpha \pm i\beta$: $\quad y_{(x)} = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$)

We have two complex roots, so the general solution of the DE is

$$y = e^{\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{7}}{14}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{14}x\right)\right] \qquad \text{for any constants } c_1 \text{ and } c_2$$

Solving a homogeneous 2nd order DE: one more example

Example: $y'' + 16y' + 64y = 0$
First construct the corresponding quadratic equation and solve it:

$$r^2 + 16r + 64 = 0$$
$$(r+8)^2 = 0$$
$$r = -8$$

We have just one root this time!
(Recall: in case there's just one root: $y_{(x)} = c_1 e^{rx} + c_2 x e^{rx}$)
Therefore the general solution is:

$$\boxed{y = c_1 e^{-8x} + c_2 \boldsymbol{x} e^{-8x}} \qquad \text{for any constants } c_1 \text{ and } c_2$$

Beware: in case there is only one root, multiply the second term (xor the first term) with x!

NON-homogeneous second order DEs

General form

$$ay'' + by' + cy = f(x)$$

(a, b and c are constant real numbers)

Plan of attack:

- Step 1: consider the complementary equation $ay'' + by' + cy = \mathbf{0}$ and compute it's solution y_c. (This is easy as it's a homogeneous equation)
- Step 2: find some particular solution y_p to the original non-homogeneous equation
- Step 3: your general solution to the original equation is now $y = y_c + y_p$

The difficulty may be mostly in step 2.

Non-homogeneous second order DEs: example 1 (part 1)

- Question: find the general solution of the differential equation $7y'' - 7y' + 2y = x^2 + 7$.
- Step 1: we had already found the complementary solution (to the equation $7y'' - 7y' + 2y = 0$) before: $y_c = e^{\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{7}}{14}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{14}x\right)\right]$ for any constants $c_{1,2}$.
- Step 2: we must find some particular solution. Since $x^2 + 7$ is a 2nd order polynomial, let's set our particular solution to $y_p = Ax^2 + Bx + C$. We plug this in the DE in order to find A, B, and C. So we have $y_p = Ax^2 + Bx + C$, $y_p' = 2Ax + B$ and $y_p'' = 2A$. Let's plug this in:

Non-homogeneous second order DEs: example 1 (part 2)

- We will plug $y_p = Ax^2 + Bx + C$, $y_p' = 2Ax + B$ and $y_p'' = 2A$ in the original DE ($7y'' - 7y' + 2y = x^2 + 7$) to find A, B and C of the particular solution:

$$7(2A) - 7(2Ax + B) + 2(Ax^2 + Bx + C) = x^2 + 7$$

$$(2A)x^2 + (-14A + 2B)x + (14A - 7B + 2C) = x^2 + 7$$

This must hold for all x, so the coefficients of the polynomials on the left- and right-hand side, must be equal. So, we have $2A = 1$, and $-14A + 2B = 0$, and $14A - 7B + 2C = 7$. From the first one, we find $A = \frac{1}{2}$, then from the second one we find $B = \frac{7}{2}$, after which the third one gives us $C = \frac{49}{4}$. Thus, we've found a particular solution: $y_p = \frac{1}{2}x^2 + \frac{7}{2}x + \frac{49}{4}$.

Non-homogeneous second order DEs: example 1 (part 3)

- Step 3: now that we have found the complementary solution $y_c = e^{\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{7}}{14}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{14}x\right)\right]$ and a particular solution $y_p = \frac{1}{2}x^2 + \frac{7}{2}x + \frac{49}{4}$, we can simply add them up to obtain the general solution of $7y'' - 7y' + 2y = x^2 + 7$:

$$\boxed{y = e^{\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{7}}{14}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{14}x\right)\right] + \frac{1}{2}x^2 + \frac{7}{2}x + \frac{49}{4}}$$

for any constants c_1 and c_2.

The method of undetermined coefficients

- In the previous example, we had $x^2 + 7$ (a polynomial of order 2) on the right-hand side of the differential equation. So we guessed that a particular solution could be a polynomial of order 2 as well ($Ax^2 + Bx + C$). In general:

Method of undetermined coefficients

We search a particular solution to the differential equation $ay'' + by' + cy = f(x)$.
Let $P_n(x)$ and $Q_n(x)$ and $R_n(x)$ denote polynomials of order n.

- If $f(x) = e^{kx} P_n(x)$, then try $y_p = e^{kx} Q_n(x)$.
- If $f(x) = e^{kx} P_n(x) \sin mx$ or $f(x) = e^{kx} P_n(x) \cos mx$, then try $y_p = e^{kx} Q_n(x) \cos mx + e^{kx} R_n(x) \sin mx$

If any term in your "guess" is a solution to the complementary equation, then multiply your guess y_p by x (or x^2 if it's still the case).
Plug your y_p-guess in the DE in order to find the coefficients of $Q_n(x)$ and $R_n(x)$.

In the previous example we had the first case (with $k = 0$ such that $e^{kx} = 1$).

The method of undetermined coefficients (examples)

- We search a particular solution to $ay'' + by' + cy = f(x)$.
- If $f(x) = x^3$ or $f(x) = 10000x^3 + x + 12$, we would try $y_p = Ax^3 + Bx^2 + Cx + D$.
- If $f(x) = \sin 8x$ or $f(x) = 137\cos 8x$, we would try $y_p = A\cos 8x + B\sin 8x$.
- If $f(x) = e^{7x}$ or $f(x) = 39e^{7x}$, we would try $y_p = Ae^{7x}$.
- If $f(x) = xe^{8x}$ or $f(x) = xe^{8x} + e^{8x}$, we would try $y_p = (Ax + B)e^{8x}$.
- If $f(x) = x^2 \sin x$, we would try $y_p = (Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x$.
- If $f(x) = e^{9x}x^2 \sin 4x$, we would try $y_p = e^{9x}(Ax^2 + Bx + C)\cos 4x + e^{9x}(Dx^2 + Ex + F)\sin 4x$.
- Notice that the last two examples are so long that you will probably not get them on your exam (since you'd have to solve for 6 coefficients). However they are useful as a demonstration of the principle.
- **Do not forget that you have to multiply your y_p-guess by x if any term in your guess is a solution to the complementary equation.**

The superposition principle

$$ay'' + by' + cy = f_1(x) + f_2(x)$$

- Sometimes, $f(x)$ is a sum of multiple functions, say $f(x) = f_1(x) + f_2(x)$. In that case, you can just find a particular solution y_{p1} to the differential equation $ay'' + by' + cy = f_1(x)$ and a particular solution y_{p2} to the differential equation $ay'' + by' + cy = f_2(x)$.
- Your particular solution to the differential equation $ay'' + by' + cy = f_1(x) + f_2(x)$ is then given by $y_{p1} + y_{p2}$.
- Do not forget to add the complementary solution to your answer as well.
- (This also works for a sum of more than two functions; see next slide for a full example.)

Superposition principle & method of u.c. (example)

- Question: solve $y'' - 6y' + 8y = xe^{3x} + xe^{4x} + xe^{5x}$.
- The complementary solution is $y_c = c_1 e^{2x} + c_2 e^{4x}$ for any constants c_1 and c_2.
- Let y_{p1} be a particular solution to $y'' - 6y' + 8y = xe^{3x}$. Then y_{p1} must be of the form $y_{p1} = (Ax + B)e^{3x}$. Substituting this in $y'' - 6y' + 8y = xe^{3x}$ gives that $A = -1$ and $B = 0$. So we find $y_{p1} = -xe^{3x}$.
- Let y_{p2} be a particular solution to $y'' - 6y' + 8y = xe^{4x}$. Then y_{p2} would be of the form $y_{p2} = (Cx + D)e^{4x}$, but we observe that the term De^{4x} is a solution to the complementary equation (since $y_c = c_1 e^{2x} + c_2 e^{4x}$), thus we multiply the y_{p2}-guess by x and obtain $y_{p2} = (Cx^2 + Dx)e^{4x}$. We substitute this into $y'' - 6y' + 8y = xe^{4x}$ and obtain $C = \frac{1}{4}$ and $D = -\frac{1}{4}$, so we find $y_{p2} = (\frac{1}{4}x^2 - \frac{1}{4}x)e^{4x}$.
- Let y_{p3} be a particular solution to $y'' - 6y' + 8y = xe^{5x}$. Then y_{p3} must be of the form $y_{p3} = (Ex + F)e^{5x}$. Substituting this in $y'' - 6y' + 8y = xe^{5x}$ gives that $E = \frac{1}{3}$ and $F = -\frac{4}{9}$. So we find $y_{p3} = (\frac{1}{3}x - \frac{4}{9})e^{5x}$.
- The particular solution to the original differential equation is now $y_{p1} + y_{p2} + y_{p3} = -xe^{3x} + (\frac{1}{4}x^2 - \frac{1}{4}x)e^{4x} + (\frac{1}{3}x - \frac{4}{9})e^{5x}$. We add the full particular solution to the complementary solution and obtain as our final answer:

$$\boxed{y = c_1 e^{2x} + c_2 e^{4x} - xe^{3x} + \left(\frac{1}{4}x^2 - \frac{1}{4}x\right) e^{4x} + \left(\frac{1}{3}x - \frac{4}{9}\right) e^{5x}} \quad \text{for all } c_1 \text{ and } c_2$$

Sample question on differential equations (slide 1)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Solution steps:

- Step 1: solve the homogeneous equation $y'' + 2y' - 35y = 0$ to find the complementary solution.
- Step 2: use the method of undetermined coefficients to find a particular solution to the original (non-homogeneous) equation.
- Step 3: we add the complementary solution to the particular solution to find the general solution of the original equation.
- Step 4: apply the initial values to obtain the final answer.
- (Fully worked out solution on next slides)

Sample question on differential equations (slide 2)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 1: first we solve $y'' + 2y' - 35y = 0$. The characteristic equation is $r^2 + 2r - 35 = 0$, thus $(r+7)(r-5) = 0$, so the roots are 5 and -7, two distinct real numbers.
Thus, the complementary solution takes the form $y_c = c_1 e^{5x} + c_2 e^{-7x}$ for any constants c_1 and c_2. (Later we will determine which c_1 and c_2 suit our initial values.)

Sample question on differential equations (slide 3)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 2: we apply the method of undetermined coefficients as explained before. $f(x) = 3e^{5x}$, so we would try the particular solution $y_p = Ae^{5x}$.

- (Recall the first case from the method of u.c.: if $f(x) = e^{kx}P_n(x)$, then try $y_p = e^{kx}Q_n(x)$. Here $P_n(x) = 3$, a "polynomial" of degree 0)

However, the complementary solution was $y_c = c_1e^{5x} + c_2e^{-7x}$ for any constants c_1 and c_2. We observe that our trial particular solution $y_p = Ae^{5x}$ will not work, because it is a solution to the complementary equation! Thus, we multiply our guess by x, so our trial particular solution is $\boldsymbol{y_p = Axe^{5x}}$, but we still need to find the constant A (next slide).

Sample question on differential equations (slide 4)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 2 (continuation): our trial particular solution is $y_p = Axe^{5x}$, but we need to find A. So we compute the derivatives: $y_p' = A(e^{5x} + 5xe^{5x})$ and $y_p'' = A(5e^{5x} + 5e^{5x} + 25xe^{5x}) = A(10e^{5x} + 25xe^{5x})$
We substitute this in the original differential equation to find:

$$A(10e^{5x} + 25xe^{5x}) + 2A(e^{5x} + 5xe^{5x}) - 35Axe^{5x} = 3e^{5x}$$
$$\iff 12Ae^{5x} = 3e^{5x}$$

So we take $A = \frac{1}{4}$. The guess worked (since we were able to find an A such that $y_p = Axe^{5x}$ satisfies the differential equation), so we found the valid particular solution $\boldsymbol{y_p = \frac{1}{4}xe^{5x}}$.

Sample question on differential equations (slide 5)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 3): the general solution to the complementary equation was $y_c = c_1 e^{5x} + c_2 e^{-7x}$ and a particular solution is $y_p = \frac{1}{4} x e^{5x}$. We add these together to obtain the general solution to the non-homogeneous (original) equation for any constants c_1 and c_2:

$$y = c_1 e^{5x} + c_2 e^{-7x} + \frac{1}{4} x e^{5x}$$

This is a solution for every c_1 and c_2, but we were given an initial value problem, i.e. we still have to find c_1 and c_2 such that $y(0) = 137$ and $y'(0) = 42$ (step 4, next slide).

Sample question on differential equations (slide 6)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 4): the general solution to the differential equation is
$y = c_1 e^{5x} + c_2 e^{-7x} + \frac{1}{4}xe^{5x}$, with derivative
$y' = 5c_1 e^{5x} - 7c_2 e^{-7x} + \frac{1}{4}e^{5x} + \frac{5}{4}xe^{5x}$.
We need to have $y(0) = 137$ and $y'(0) = 42$, i.e.

$$y(0) = c_1 + c_2 = 137 \qquad y'(0) = 5c_1 - 7c_2 + \frac{1}{4} = 42$$

Substituting $c_2 = 137 - c_1$ into the second equation gives
$5c_1 - 7(137 - c_1) + \frac{1}{4} = 42 \iff 12c_1 = \frac{4003}{4} \iff c_1 = \frac{4003}{48}$ and from we first equation we obtain $c_2 = 137 - \frac{4003}{48} = \frac{2573}{48}$. Therefore, the solution is

$$\boxed{y = \frac{4003}{48}e^{5x} + \frac{2573}{48}e^{-7x} + \frac{1}{4}xe^{5x}}$$

Method of variation of constants

We already discussed the method of undetermined coefficients to solve nonhomogeneous linear ODEs. Now we discuss another method: **variation of constants** (or **variation of parameters**). Suppose we have the general solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$ to some *homogeneous* DE $ay'' + by' + cy = 0$.

- Then we can replace the constants c_1, c_2 by functions $u_1(x)$ and $u_2(x)$, and try to find a *particular* solution to the *nonhomogeneous* equation $ay'' + by' + cy = f(x)$ of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

- Then we solve this system of equations for u_1' and u_2':

$$\begin{cases} a(u_1'y_1' + u_2'y_2') = f & \text{(this comes from subbing } y_p \text{ into the DE)} \\ u_1'y_1 + u_2'y_2 = 0 & \text{(this is an additional constraint we impose!)} \end{cases}$$

- Then we integrate u_1' and u_2' to find u_1 and u_2, and then we found the general solution to the nonhomogeneous equation.
- Note: this method can be extended to many more types of equations, e.g. equations with nonconstant coefficients. This is helpful if you have to find the general solution given some particular solutions.

Variation of constants: condensed example

Question: solve $y'' + 3y' + 2y = \sin(e^x)$.
Solution: a general solution to the equation $y'' + 3y' + 2y = 0$ is given by $y(x) = c_1 e^{-x} + c_2 e^{-2x}$. Now we have to solve the system of equations

$$\begin{cases} -u_1' e^{-x} - 2u_2' e^{-2x} = \sin(e^x) & \text{(note that } a = 1\text{)} \\ u_1' e^{-x} + u_2' e^{-2x} = 0 \end{cases}$$

Adding the two rows together, we find $-u_2' e^{-2x} = \sin(e^x)$, so $u_2' = -e^{2x}\sin(e^x)$.
Then, we can also find $u_1' = e^x \sin(e^x)$.
Both integrals can be solved by substituting $t = e^x$ (then $dt = e^x dx$):

$$u_1(x) = -\cos(e^x)$$
$$u_2(x) = e^x\cos(e^x) - \sin(e^x) \qquad \text{(omitting constants of integration)}$$

So a particular solution to the nonhomogeneous equation is $y_p(x) = -\cos(e^x)e^{-x} + (e^x\cos(e^x) - \sin(e^x))e^{-2x} = -e^{-2x}\sin(e^x)$. So the general solution is

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x}\sin(e^x)}$$

Extra: constructing an ODE from its solutions

- **Question**: give a linear homogeneous ODE with constant coefficients of minimal order, which admits the solutions $y_1 = 42\cos x$, $y_2 = -x\cos x$ and $y_3 = e^{42x}$.
- **Solution**: think backwards. If y_1, y_2 and y_3 are solutions, they must correspond to roots in the characteristic equation of the ODE.
- The solution $42\cos x$ corresponds to the roots $r = \pm i$. As $-x\cos x$ must also be a solution, these roots must both occur with multiplicity two.
- The solution e^{42x} corresponds to the root $r = 42$.
- So the characteristic equation must be

$$(r^2+1)^2(r-42) = 0$$

- Expanding brackets and replacing powers of r by derivatives of y, we find the answer

$$\boxed{y''''' - 42y'''' + 2y''' - 84y'' + y' - 42y = 0}$$

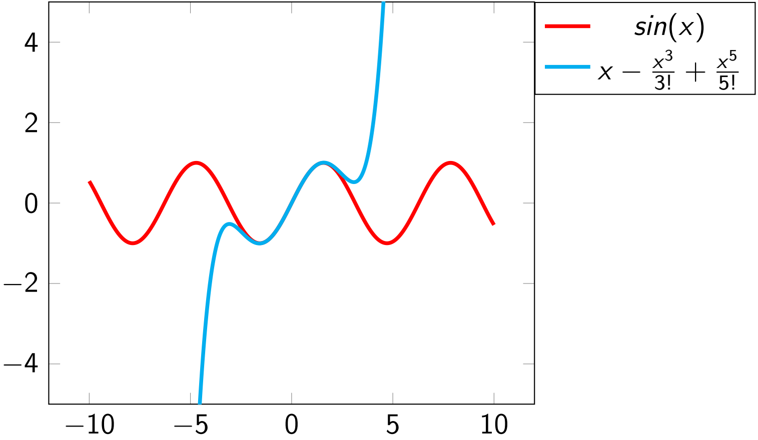
Taylor polynomials

Sometimes, we want to approximate a function by a polynomial.
The nth degree *Taylor polynomial* of function f at $x = a$ is given by

$$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j,$$

provided f is differentiable n times.
This definition has been chosen such that the ith derivatives of f are equal to the ith derivatives of $T_n(x)$ for $i \in \{0, 1, \ldots, n\}$. Intuitively, this ensures that around $x = a$, the behavior of the polynomial is similar to the behavior of f, i.e., the polynomial *approximates* f.

Visualization



Interactive version: `https://www.desmos.com/calculator/elb2sjyuhu`

Example Taylor polynomial question

Find an expression for the $(2n+1)$th degree Taylor polynomial $T_{2n+1}(x)$ of the function $f(x) = \sin x$ centered around $x = 0$.

Solution: let's compute some derivatives:

$$\begin{aligned} f^{(0)}(x) = f(x) = \sin x \qquad & f^{(0)}(0) = 0 \\ f^{(1)}(x) = \cos x \qquad & f^{(1)}(0) = 1 \\ f^{(2)}(x) = -\sin x \qquad & f^{(2)}(0) = 0 \\ f^{(3)}(x) = -\cos x \qquad & f^{(3)}(0) = -1 \\ f^{(4)}(x) = \sin x \qquad & f^{(4)}(0) = 0 \\ f^{(5)}(x) = \cos x \qquad & f^{(5)}(0) = 1 \end{aligned}$$

We see a pattern! These derivatives will infinitely repeat in a cycle of length four. Using the definition, we get

$$T_{2n+1}(x) = \sum_{j=0}^{2n+1} \frac{f^{(j)}(a)}{j!}(x-a)^j = \sum_{j=0}^{2n+1} \frac{f^{(j)}(0)}{j!} x^j$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \boxed{\sum_{i=0}^{n} \frac{(-1)^i x^{2i+1}}{(2i+1)!}}.$$

Evaluating limits with Taylor polynomials

We can do some nice things with Taylor polynomials, such as evaluating limits.

- **Question**: evaluate $\lim_{x\to 0} \frac{(\sin\sqrt{x})^2 - x\cos x + \frac{1}{3}x^2}{x\ln(1+x) - x^2}$.
- **Solution**: replace some of the terms by their Taylor polynomial:

$$\lim_{x\to 0} \frac{(\sin\sqrt{x})^2 - x\cos x + \frac{1}{3}x^2}{x\ln(1+x) - x^2}$$

$$= \lim_{x\to 0} \frac{\left[x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} + O(x^{7/2})\right]^2 - x\left[1 - \frac{x^2}{2!} + O(x^4)\right] + \frac{1}{3}x^2}{x\left[x - \frac{x^2}{2} + O(x^3)\right] - x^2}$$

$$= \lim_{x\to 0} \frac{\left[x - \frac{1}{3}x^2 + (\frac{1}{3!3!} + \frac{2}{5!})x^3 + O(x^4)\right] - \left[x - \frac{x^3}{2!} + O(x^5)\right] + \frac{1}{3}x^2}{\left[x^2 - \frac{x^3}{2} + O(x^4)\right] - x^2}$$

$$= \lim_{x\to 0} \frac{(\frac{1}{3!3!} + \frac{2}{5!} + \frac{1}{2!})x^3 + O(x^4)}{-\frac{1}{2}x^3 + O(x^4)} = \lim_{x\to 0} \frac{\frac{49}{90} + O(x)}{-\frac{1}{2} + O(x)} = \frac{\frac{49}{90} + 0}{-\frac{1}{2} + 0} = \boxed{-\frac{49}{45}}.$$

Note: in a similar fashion, you can approximate integrals using Taylor polynomials.

Taylor's theorem

Let us write $f(x) = T_n(x) + E_n(x)$, where $T_n(x)$ is the nth degree Taylor polynomial of the function f around $x = a$. We call $E_n(x)$ the *error term*.

Taylor's theorem

If f is $n+1$ times differentiable on the open interval between a and x and $f^{(n)}$ is continuous on the closed interval between a and x, then

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Error estimation theorem (corollary): if there is a positive constant M for which $|f^{(n+1)}(y)| \leq M$ for all y between a and x, then

$$|E_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}.$$

Error estimation using Taylor's theorem

Question: estimate $\sin(1^\circ)$ with an error less than 10^{-13}.

Solution:

- We want to approximate $\sin(1^\circ) = \sin\frac{\pi}{180}$ using the Taylor polynomials of $\sin x$ (centered around $x = 0$). So, for which n do we have $|E_n(\frac{\pi}{180})| < 10^{-13}$?
- By the Error estimation theorem with $M = 1$ (why?), we find that $|E_n(\frac{\pi}{180})| < 10^{-13}$ whenever $\frac{(\frac{\pi}{180})^{n+1}}{(n+1)!} < 10^{-13}$. This holds for $n \geq 5$.
- Hence, it suffices to use the Taylor polynomial of degree 5. We get

$$\sin(1^\circ) \approx (\pi/180) - \frac{(\pi/180)^3}{6} + \frac{(\pi/180)^5}{120} = 0.0174524064372836107\ldots$$

- Verification: $\sin(\frac{\pi}{180}) = 0.0174524064372835128\ldots$
  Verdict: success!